Finite Difference Solutions of the Heat Equation in a Molten Polymer Flowing in a Circular Tube

C. H. Leong,[1] D. G. Mohr,[2] M. B. M. Elgindi,[2] R. W. Langer[2]

[1]*Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455*
[2]*Department of Mathematics, University of Wisconsin, Eau Claire, Wisconsin 54702-4004*

Received 9 August 2005; accepted 24 October 2005
DOI 10.1002/app.23780
Published online in Wiley InterScience (www.interscience.wiley.com).

ABSTRACT: We present the mathematical equations that govern heat transfer in a polymer melt flowing in a circular tube with constant ambient temperature, taking into account the viscous dissipation effects. This leads to a nonlinear parabolic partial differential equation. It is shown that the exact solution of a linearized version of the governing equation can be presented in terms of the Whittaker function. A finite difference scheme is used to produce an approximate solution of the linearized problem. This numerical solution is shown to be a good approximation to the exact solution found in terms of the Whittaker function. The finite difference scheme is then modified to approximate the nonlinear parabolic partial differential equation and is compared with the results found using the finite element method. © 2006 Wiley Periodicals, Inc. J Appl Polym Sci 102: 289–294, 2006

Key words: modeling; thermodynamics; rheology; calculations

INTRODUCTION

The continuity equation for a fluid flowing through a circular tube of radius R and length L is given in cylindrical coordinates (r, θ, x) by

$$\frac{\partial \rho}{\partial t} = -\left[\frac{1}{r}\frac{\partial}{\partial r} r\rho V_r + \frac{1}{r}\frac{\partial}{\partial \theta}\rho V_\theta + \frac{\partial}{\partial x}\rho V_x\right], \quad (1)$$

$0 \le r \le R$, $0 \le \theta \le 2\pi$, $0 \le x \le L$, where ρ is the fluid density and V_r, $V\theta$, and V_x are the components of velocity in the coordinate directions. For an incompressible steady-state flow, ρ is constant and $V\theta = V_r = 0$, and eq. (1) reduces to

$$\frac{\partial}{\partial x} V_x = 0. \quad (2)$$

The momentum flux balance on a fluid element is given by

$$\rho\frac{\partial V_x}{\partial t} = -\rho\left(V_r\frac{\partial V_x}{\partial r} + \frac{V_0}{r}\frac{\partial V_x}{\partial \theta} + V_x\frac{\partial V_x}{\partial x}\right) - \left(\frac{1}{r}\frac{\partial}{\partial r} r\tau_{rx} + \frac{1}{r}\frac{\partial}{\partial \theta}\tau\theta x + \frac{\partial}{x}\tau_{xx}\right) - \frac{\partial P}{\partial x} + \rho g_x \quad (3)$$

where P is the fluid pressure, ρg_x is the gravitational force per unit volume, and τ is the shear stress.[1] The first subscript of τ indicates the axis perpendicular to the plane on which the stress is acting and the second subscript indicates the direction of the stress. The gravitational term will be neglected, as it is relatively small in comparison to the frictional forces and pressure. It follows from the steady-state assumption that τ_{rx} is the only nonzero component of shear stress. Thus, eq. (3) reduces to

$$0 = -\frac{1}{r}\frac{\partial}{\partial r}\tau\tau_{rx} - \frac{\partial P}{\partial x}. \quad (4)$$

For a nonisothermal flow, the energy equation must also be considered. The energy equation is given by Ref. 1.

$$\rho C_v\left(\frac{\partial T}{\partial t} + V_r\frac{\partial T}{\partial r} + V_\theta\frac{\partial T}{\partial \theta} + V_x\frac{\partial T}{\partial x}\right) = -\left(\frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{1}{r}\frac{\partial q\theta}{\partial \theta} + \frac{\partial q_r}{\partial_x}\right) - \left(\frac{\partial \ln\rho}{\partial \ln t}\right)_p\frac{Dp}{Dt} - (\tau : \nabla v) \quad (5)$$

where T is temperature, C_v is the heat capacity, q_r, $q\theta$, and q_x are the heat flux in the r, θ, and x directions, respectively, and $(\tau : \nabla v)$ is the viscous dissipation term defined by

Correspondence to: M. B. M. Elgindi (elgindmb@uwec.edu).
Contract grant sponsors: University of Wisconsin System Applied Research Grant 2004–2005; University of Wisconsin-Eau Claire Office of University Research and Sponsored Programs; Extrusion Dies Industries, LLC, Chippewa Falls, Wisconsin.

Journal of Applied Polymer Science, Vol. 102, 289–294 (2006)

$$(\tau:\nabla v) = -\left[\tau_{rr}\frac{\partial V_r}{\partial_r} + \tau_{\theta\theta}\frac{1}{T}\left(\frac{\partial V_\theta}{\partial\theta} + V_r\right) + \tau_{xx}\frac{\partial V_x}{\partial_x}\right] - \left[\tau_{x\theta}\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right] + \tau_{rx}\left(\frac{\partial V_x}{\partial_r} + \frac{\partial V_r}{\partial x}\right) + \tau_{\theta x}\left(\frac{1}{r}\frac{\partial V_x}{\partial\theta} + \frac{\partial V_\theta}{\partial x}\right)\right].$$

The steady-state criterion implies $\frac{\partial T}{\partial t} = 0$. Using the conditions $V_r\frac{\partial T}{\partial r} = 0$, $V_\theta\frac{\partial T}{\partial\theta} = 0$, and $(\tau : \nabla v) = \tau_{rx}\frac{\partial V_x}{\partial_r}$ and assuming that the heat flux in the θ direction is 0, eq. (5) reduces to

$$\rho C_v V_x\frac{\partial T}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial_r}(rq_r) - \frac{\partial q_x}{\partial_x} - \tau_{rx}\frac{\partial V_x}{\partial_r}. \tag{6}$$

Using Fourier's law of heat conduction, $q_x = -k\frac{\partial T}{\partial_x}$ and $q_r = -k\frac{\partial T}{\partial_r}$, where k is the thermal conductivity of the pipe, eq. (6) becomes

$$\rho C_v V_x\frac{\partial T}{\partial_x} = \frac{k}{r}\frac{\partial}{\partial_r}\left(r\frac{\partial T}{\partial_r}\right) + k\frac{\partial^2 T}{\partial_x{}^2} - \tau_{rx}\frac{\partial V_x}{\partial_r}. \tag{7}$$

Using $\tau_{rx} = \eta\gamma$, where η is the viscosity and γ is the shear rate, $\gamma = -\frac{\partial V_x}{\partial_r}$, eq. (7) reduces to

$$\rho C_v V_x\frac{\partial T}{\partial_x} = k\frac{\partial^2 T}{\partial_r{}^2} + \frac{k}{r}\frac{\partial T}{\partial r} + k\partial\frac{\partial^2 T}{\partial_x{}^2} - \tau_{rx}\frac{\partial V_x}{\partial_r}. \tag{8}$$

For a power law fluid, the viscosity η can be described by

$$\eta = \lambda\dot{\gamma}^{n-1}, \tag{9}$$

where λ is the consistency factor and n is the power law exponent. For most polymers, n ranges from 0.2 to 0.7 with $n = 1$ corresponding to Newtonian behavior.[2] In general, viscosity is a function of shear rate, temperature, pressure, and time. In this article, we will only discuss the case where viscosity is dependent on temperature and shear rate.

If $\eta(\dot{\gamma}, T_0)$ and $\eta(\dot{\gamma}, T)$ are viscosity functions for the same polymer at temperatures T_0 and T, respectively, the Arrhenius law states that $\eta(\dot{\gamma}, T_0)$ and $\eta(\dot{\gamma}, T)$ are related by

$$\eta(\dot{\gamma},T) = \eta(\dot{\gamma},T_0)e^{\alpha(T_0-T)}, \tag{10}$$

where α is a material dependent constant. Using eq. (9), eq. (10) becomes

$$\eta(\dot{\gamma},T) = \lambda\dot{\gamma}^{n-1}e^{\alpha(T_0-T)}, \tag{11}$$

where $\eta(\dot{\gamma}, T_0) = \lambda\dot{\gamma}^{n-1}$. Substituting (11) in (8) yields

$$\rho C_v V_x\frac{\partial T}{\partial_x} = k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r}\frac{\partial T}{\partial r} + k\frac{\partial^2 T}{\partial_x{}^2} + \lambda\dot{\gamma}^{n+1}e^{\alpha(T_0-T)}. \tag{12}$$

V_x and $\dot{\gamma}$ can be written in terms of the average flow velocity $\bar{V}_x$ as follows.[2]

$$V_x = \bar{V}_x\frac{m+3}{m+1}\left[1 - \left(\frac{r}{R}\right)^{m+1}\right] \tag{13}$$

$$\dot{\gamma} = (m+3)\frac{V_x}{R}\left(\frac{r}{R}\right)^m \tag{14}$$

Substituting eqs. (13) and (14) into eq. (12), we get

$$\rho C_v\bar{V}_x\frac{m+3}{m+1}\left[1 - \left(\frac{r}{R}\right)^{m+1}\right]\frac{\partial T}{\partial_x} = k\frac{\partial^2 T}{\partial_r{}^2} + \frac{k}{r}\frac{\partial T}{\partial_r} + k\frac{\partial^2 T}{\partial_x{}^2} + \lambda\frac{(m+3)^{n+1}\bar{V}_x{}^{n+1}}{R^{n+1}}\left(\frac{r}{R}\right)^{m(n+1)}e^{\alpha(T_0-T)} \tag{15}$$

This partial differential equation describes the temperature in the flow channel as a function of r and x. The boundary conditions are:

$$\begin{aligned}
&\text{at } r=0, \text{ the temperature is finite,}\\
&\text{at } r = R,\ k\frac{\partial T}{\partial r} = q_0 \text{ (constant heat flux at the wall),}\\
&\text{at } x = 0,\ T = T_1 \text{ (uniform inlet fluid temperature),}\\
&\text{at } r = 0,\ \frac{\partial T}{\partial_r} = 0 \text{ (symmetry).}
\end{aligned} \tag{16}$$

We introduce the dimensionless variables

$$\bar{r} = \frac{r}{R},$$

$$u = \frac{(m+1)k}{(m+3)\rho C_v\overline{V_x}R^2}x,$$

$$A = \lambda e^{\alpha T_0}\frac{(M+3)^{n=1}\overline{V_x}^{n=1}}{kR^{n-1}}.$$

With these dimensionless variables, eq. (15) becomes the nonlinear elliptic partial differential equation

$$[1 - (\bar{r})^{m+1}]\frac{\partial T}{\partial_u} = \frac{\partial^2 T}{\partial\bar{r}^2} + \frac{1}{r}\frac{\partial T}{\partial\bar{r}} + AR^2e^{-\alpha T}\bar{r}^{m+1} \tag{17}$$

where

$$D = \left[\frac{(m+1)k}{(m+3)\rho C_v \bar{V}_x R^2}\right]^2.$$

When k is small, D is small and the term containing D in eq. (17) may be neglected. In this case, eq. (17) reduces to the parabolic equation

$$[1-(\bar{r})^{m+1}]\frac{\partial T}{\partial_u} = \frac{\partial^2 T}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial T}{\partial \bar{r}} + AR^2 e^{-\alpha T}\bar{r}^{m+1} \quad (18)$$

Remark: eq. (18) approximates the flow away from the entrance of the pipe where the term $\frac{\partial^2 T}{\partial_u{}^2}$ is small. Near the entrance, the term containing D in eq. (17) may be large and cannot be neglected. Therefore, the results of this article will represent the flow well far from the entrance.

TEMPERATURE PROFILE FOR NEWTONIAN FLOW

For Newtonian flow, $m = 1$. Assuming k is very small, we may neglect the term containing D as well as the viscous term, and eq. (18) becomes

$$[1-(\bar{r})^2]\frac{\partial T}{\partial_u} = \frac{\partial^2 T}{\partial \bar{r}^2} + \frac{1}{r}\frac{\partial T}{\partial \bar{r}}. \quad (19)$$

Equation (19) is linear and can be solved analytically. with the change of variable

$$\Theta = \frac{k(T-T_0)}{q0R},$$

Equation (19) becomes

$$[1-(\bar{r})^2]\frac{\partial \Theta}{\partial u} = \frac{\partial^2 \Theta}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \Theta}{\partial \bar{r}}. \quad (20)$$

Following Ref. 1, we assume a solution for eq. (20) of the form

$$\Theta(u,\bar{r}) = \Theta_\infty(u,\bar{r}) - \Theta_d(u,\bar{r}) \quad (21)$$

where Θ_∞ $(u, \bar{r})$ is the asymptotic solution and is given by

$$\Theta_\infty(u,\bar{r}) = 4u + \bar{r}^2 - \frac{1}{4}\bar{r}^4 - \frac{7}{24},$$

and Θ_d is expected to dampen exponentially with time. By substituting eq. (21) into eq. (20), we find that Θ_d $(u, \bar{r})$ must satisfy the following boundary conditions:

$$\begin{aligned} &\text{at } \bar{r} = 0, \frac{\partial \Theta}{\partial \bar{r}} = 0,\\ &\text{at } \bar{r} = 1, \frac{\partial}{\partial \Theta \bar{r}} = 0,\\ &\text{at } u = 0, \Theta_d = \Theta_\infty(0,\bar{r}). \end{aligned} \quad (22)$$

If we assume that Θ_d $(u, \bar{r})$ is separable, that is

$$\Theta_d(u,\bar{r}) = Z(u)X(\bar{r}), \quad (23)$$

then, eqs. (20) and (21) can be separated into the differential equations:

$$\frac{\partial Z}{\partial u} = -c^2 Z, \quad (24)$$

$$\frac{\partial^2 X}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial X}{\partial \bar{r}} + c^2(1-\bar{r}^2)X = 0, \quad (25)$$

where $-c^2$ is the separation constant. Equation (25) is a Sturm-Liouville problem and has an infinite number of eigenvalues c_k and eigenfunctions X_k. Thus, Θ_d $(\bar{r}, u)$ must be of the form

$$\Theta_d(u,\bar{r}) = \sum_{k=1}^{\infty} B_k e^{-c_k^2 u} X_k(\bar{r}).$$

From the Sturm-Liouville theory,

$$B_k = \frac{\int_0^1 \Theta_\infty(0,\bar{r})X_k(\bar{r})(1-\bar{r}^2)\bar{r}d\bar{r}}{\int_0^1 [X_k(\bar{r})]^2(1-\bar{r}^2)\bar{r}d\bar{r}},$$

and the eigenfunctions X_k can be found by solving eq. (25).[3]

To solve (25), let $X_k = \frac{V_k}{\bar{r}}$. Then,

$$\frac{\partial X_k}{\partial \bar{r}} = \frac{1}{\bar{r}}\frac{\partial V_k}{\partial \bar{r}} - \frac{V_k}{\bar{r}^2}, \quad (26)$$

$$\frac{\partial^2 X_k}{\partial \bar{r}^2} = \frac{1}{r}\frac{\partial^2 V_k}{\partial \bar{r}^2} - \frac{2}{r^2}\frac{\partial V_k}{\partial \bar{r}} + \frac{2V_k}{\bar{r}^3}. \quad (27)$$

Substituting eqs. (26) and (27) into eq. (25) and rearranging terms, we obtain

$$\frac{\partial^2 V_k}{\partial \bar{r}^2} - \frac{1}{r}\frac{\partial V_k}{\partial \bar{r}} + \frac{V_k}{\bar{r}^2} + c_k^2(1-\bar{r}^2)V_k = 0. \quad (28)$$

With the change of variables $y = c_k\bar{r}^2$

$$\frac{\partial V_k}{\partial \bar{r}} = 2\sqrt{c_k y}\,\frac{\partial V_k}{\partial_y}, \tag{29}$$

$$\frac{\partial^2 V_k}{\partial \bar{r}^2} = 4c_k y\,\frac{\partial^2 V_k}{\partial_y{}^2} + 2c_k\,\frac{\partial V}{\partial y}. \tag{30}$$

Substituting eqs. (29) and (30) into eq. (28), we get

$$\frac{\partial^2 V_k}{\partial y^2} + \left(-\frac{1}{4} + \frac{c_k}{4y} + \frac{4}{y^2}\right)V_k = 0. \tag{31}$$

Equation (31) is a Whittaker differential equation whose solutions $M_{k,m'}(y)$ and $W_{k,m}(y)$ are given by

$$M_{k,m}(y) = e^{\frac{-y}{2}} y^{m+\frac{1}{2}} F_{1,1}\left(\frac{1}{2} + m - k, 1 + 2m; y\right),$$

$$W_{k,m}(y) = e^{\frac{-y}{2}} y^{m+\frac{1}{2}} U\left(\frac{1}{2} + m - k, 1 + 2m; y\right),$$

where $F_{1,1}$ $(a, b; y)$ and $U(a, b; y)$ are hypergeometric functions. Therefore, we can write $V_k(y)$ as a linear combination of $M_{k,m}(y)$ and $W_{k,m}(y)$,

$$V_k(y) = SM_{\frac{c_k}{4},0}(y) + QW_{\frac{c_k}{4},0}(y),$$

The boundary conditions on the eigenfunctions are

$$\begin{aligned} &\text{at } \bar{r} = 0, \frac{\partial X_k(\bar{r})}{\partial \bar{r}} = 0, \\ &\text{at } \bar{r} = 1, \frac{\partial X_k(\bar{r})}{\partial \bar{r}} = 0. \end{aligned} \tag{32}$$

Using these boundary conditions, we find that Q must be 0, and $S = \frac{1}{\sqrt{c_k}} C_k$. Thus, the eigenfunctions are given by

$$X_k(\bar{r}) = \frac{1}{\sqrt{c_k}}\; M_{\frac{c_k}{4},0}(c_k \bar{r}^2) = e^{-\frac{c_k}{2}r^2} F_{1,1}\left(\frac{1}{2} - \frac{c_k}{4}, 1; c_k \bar{r}^2\right).$$

The c_k's can be found by applying the second boundary condition of eq. (32). Each corresponding X_k is evaluated and used to find B_k. The first 10 values for each are shown in Table I. Our results compare well with the results of Ref. 4.

TABLE I
The First 10 Values for c_k, X_k, and B_k

k	c_k	X_k (1)	B_k
1	5.067505501	−0.4925165736	0.4034832179
2	9.157606426	0.3955084753	−0.175110001
3	13.19722474	−0.3458736768	0.10559172
4	17.22022936	0.3140464817	−0.073282404
5	21.23551728	−0.2912514573	0.055036507
6	25.24653118	0.2738069527	−0.043484384
7	29.25490555	−0.2598530271	0.03559511
8	33.26152373	0.2483319661	−0.02845542
9	37.26690821	−0.2385903994	0.025640121
10	41.27138935	0.2301992523	−0.022333705

FINITE DIFFERENCE METHOD

In this section, we use an explicit finite difference scheme to approximate the solution of eq. (20). The domain of $\theta(\bar{r}, u)$ is replaced by a grid. At each point on the grid, eq. (20) is represented by a difference equation. The values of $\theta(\bar{r}, u)$ at the boundary points of the grid are given from the boundary conditions, and $\theta(\bar{r}, u)$ at the remaining points of the grid are determined using the difference equations.

Using the difference quotients,

$$\frac{\partial \Theta(u,\bar{r})}{\partial u}\Big|_{(u_k,r_n)} \approx \frac{\Theta(u_{k+1} + \Delta u, \bar{r}_n) - \Theta(u_k, \bar{r}_n)}{\Delta u} = \frac{\Theta_{k+1,n} - \Theta_{k,n}}{\Delta u},$$

$$\frac{\partial \Theta(u,\bar{r})}{\partial \bar{r}}\Big|_{(u_k,r_n)} \approx \frac{\Theta(u_k, \bar{r}_n + \Delta \bar{r}) - \Theta(u_k, \bar{r}_n)}{\Delta \bar{r}} = \frac{\Theta_{k,n+1} - \Theta_{k,n}}{\Delta \bar{r}},$$

$$\frac{\partial^2 \Theta(u,\bar{r})}{\partial \bar{r}^2}\Big|_{(u_k,r_n)} \approx \frac{\Theta(u_k, \bar{r}_n + \Delta \bar{r}) - 2\Theta(u_k, \bar{r}_n) + \Theta(u_k, \bar{r}_n - \Delta \bar{r})}{(\Delta \bar{r})^2} = \frac{\Theta_{k,n+1} - 2\Theta_{k,n} + \Theta_{k,n-1}}{(\Delta \bar{r})^2},$$

Eq. (20) becomes

$$\Theta_{k+1,n} = \Theta_{k,n} + \frac{\Delta u}{1 - \bar{r}^2}\left[\frac{\Theta_{k,n+1} - 2\Theta_{k,n} + \Theta_{k,n-1}}{(\Delta r)^2} + \frac{1}{\bar{r}}\,\frac{\Theta_{k,n} - \Theta_{k,n-1}}{\Delta \bar{r}}\right]. \tag{33}$$

Equation (33), together with the following boundary conditions,

$$\Theta_{0,n} = 0 \text{ for } 0 \le n \le NT,$$

$$\frac{\partial \Theta(u,\bar{r})}{\partial \bar{r}}\Big|_{r=0} = 0 \text{ (symmetry)},$$

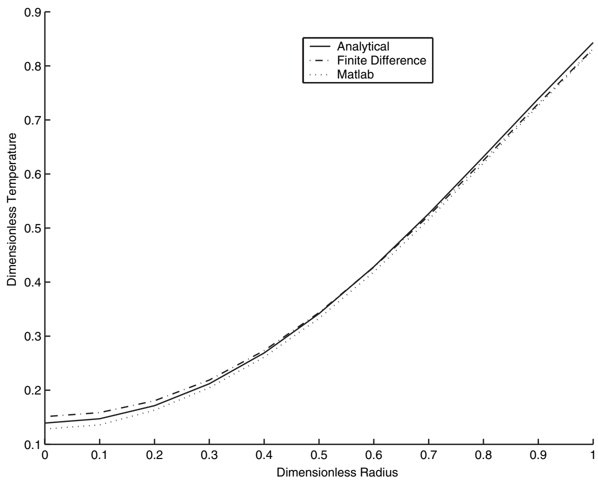

Figure 1 Temperature profile of Newtonian fluid ($u = 0.1$).

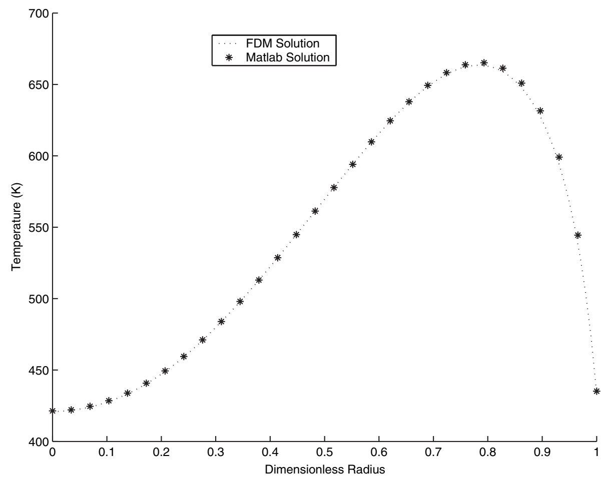

Figure 2 Temperature profile comparison at $u = 0.017$ axial distance.

is used to approximate $\theta(u, \bar{r})$ at the remaining points of the grid. Notice that a weighted average is used at the corner of the grid. Notice also that by applying the limits as $\bar{r}$ approaches 1 from below and using the condition (32), eq. (33) may be replaced by $\Theta_{k+1,0} = \Theta_{k,0} + 4\Delta u \dfrac{\Theta_{k,1} - \Theta_{k,0}}{(\Delta\bar{r})^2}$ for $\bar{r} = 1$. In Figure 1, we compare the numerical results obtained using the finite difference method and Matlab's pde solving package with the analytical solution at 0.1 unit distance from the entrance of the pipe.

From Figure 1, we see that the finite difference scheme developed here provides a reasonable approximation to the analytical solution. To ensure stability of the finite difference solution, we used $\Delta x \leq (\Delta r)^2$.[5]

NON-NEWTONIAN FLOW

In this section, we combine the finite difference scheme developed in the previous section with the fixed point iteration to approximate the solution of the nonlinear eq. (17). The temperature value $\Theta_{k,n}$ is used as a starting value in the viscous term for $\Theta_{k+1,n}$ in eq. (18). Iteration is then implemented until the difference between two successive terms is less than a specified tolerance. We use the following data taken from Ref. 6: $T_0 = 399.5$ K, $T_1 = 403.15$ K, $T_w = 433.15$ K, $V_x = 15$ cm/s, $n = 0.453$, $\rho = 0.000794$ km/cm^3, $C_v = 2510$ J/(kg K), $k = 0.00255$ W/(cm K), $R = 0.125$ cm, $\lambda = 2.82$ Ns^n/cm^2, and $\alpha = 0.01872$ K^{-1}. Note that with these data, $D = 1.7667 \times 10^{-7}$, and the heat conductive term $D\dfrac{\partial^2 T}{\partial u^2}$ of eq. (17) can be neglected. Figure 2 shows the finite difference solution of eq. (18), along with the solution obtained using Matlab's pde solving tools.

Numerical steady state, when using the finite difference method, is achieved at approximately $u = 1.85$. After this, there is no change in the temperature profile up to the fifth decimal position. At steady state, eq. (18) can be solved analytically.[7] The steady-state solution is given by

$$T(u,\bar{r}) = T_w + \frac{2}{\alpha}\ln\frac{C_1\bar{r}^{\frac{3n+1}{n}} + 1}{C_1 + 1} \tag{34}$$

where

$$C_1 = \sqrt{\left(\frac{nBC + \left(\frac{3n+1}{n}\right)^2 e^{nBT_w}}{nBC}\right)^2 - 1} - \frac{nBC + \left(\frac{3n+1}{n}\right)^2 e^{nBT_w}}{nBC},$$

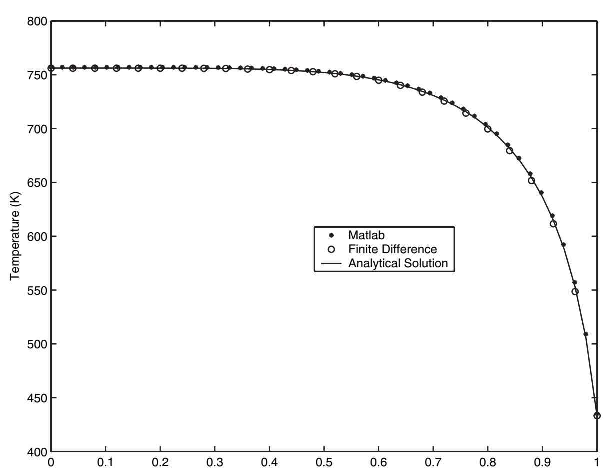

Figure 3 Numerical solutions ($u = 1.85$) *versus* the analytical solution at steady state.

$$C = \frac{\bar{V}_x^{n+1}}{k} W \, e^{nBT_0} \frac{\left(\frac{3n+1}{n} + 2\right)^2}{R^{n-1}}$$

and $W = 2.82$ Ns^n/cm^2.[6] In Figure 3, we compare the numerical solutions obtained using the finite difference method and using the Matlab pde solver with the analytical solution (34).

Bulk temperature is often calculated to find an average temperature in a circular cross section of the pipe. This quantity is defined by

$$T_{Bulk}(u) = \frac{\int_0^1 T(u,\bar{r})Y(u,\bar{r})\bar{r}d\bar{r}}{\int_0^1 Y(u,\bar{r})\bar{r}d\bar{r}}, \tag{35}$$

where $Y(u, \bar{r})$ is the velocity function and is given by

$$Y(u,\bar{r}) = \bar{V}_x \frac{(m+3)(1-r^{m+1})}{(m+1)}.$$

Wei and Zhang found that $\lim_{u\to\infty} T_{\mathrm{Bulk}}(u) = 725.55$ K.[7] Combining the finite difference method introduced here and Simpson's method, we obtain $\lim_{u\to\infty} T_{\mathrm{Bulk}}(u) = 730.19$ K, while Matlab yields $\lim_{u\to\infty} T_{\mathrm{Bulk}}(u) = 731.20$ K.

Figure 4 compares the numerical results obtained using the finite difference method and Matlab pde solver.

In conclusion, we observe that the finite difference scheme developed in this article has the advantage of being simpler than the methods used in Refs. 7 and 8, yet gives as accurate numerical results.

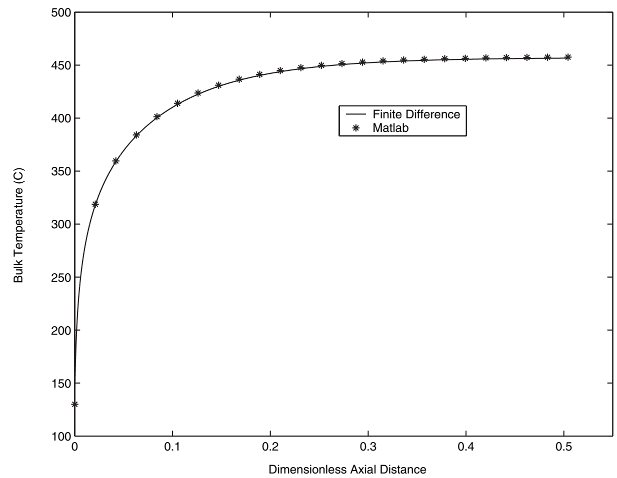

Figure 4 Bulk temperature throughout a 150 cm pipe (u = 0.50).

The authors thank Professor Dongming Wei for his comments on this article. The authors also thank the referee of this article for the useful comments.

References

1. Bird, R. B.; Stewart, W. E.; Lightfoot, E. N. Transport Phenomena; Wiley: New York, 2002.
2. Michaeli, W. Extrusion Dies for Plastics and Rubber; Hanser: Munich, 1992.
3. Zauderer, E. Partial Differential Equations of Applied Mathematics; Wiley: New York, 1989.
4. Siegel, R.; Sparrow, E. M.; Hallman, T. M. J Appl Sci Res A 1958, 7, 386.
5. Mitchell, A. R.; Griffiths, D. F. The Finite Difference Method in Partial Differential Equations; Wiley: New York, 1990.
6. Agur, E. E.; Vlachopoulos, J. J Appl Polym Sci 1981, 26, 765.
7. Wei, D.; Zhang, Z. Electron J Differ Equat 2001, 2001, 1.
8. Wei, D.; Luo, H. Int J Heat Mass Transfer 2003, 46, 3097.